Math 10B with Professor Stankova
Worksheet, Discussion #7; Thursday, 2/14/2019
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1 Induction

1.1 Concepts

1. Mathematical induction allows us to prove a statement for all n. Each induction problem will be of the form: "Let S_n be the statement that (something) is true for any integers $n \geq 1$" where (something) is some mathematical equality. To solve them, there are three steps:

    1. Base Case: Show that the statement is true for the smallest value $n = 1$.
    2. Inductive Step: State that you are assuming the inductive hypothesis (S_n is true for **some** $n \geq 1$). Then, prove that S_{n+1} is true using S_n.
    3. Conclusion: State that by MMI, we conclude that S_n is true **for all** $n \geq 1$.

    All steps must be written in order to get full credit.

1.2 Examples

2. Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Solution: First we show the base case of $n = 1$. In that case, we have that $1 = \frac{1(2)}{2}$ as required. Now assume the inductive hypothesis $S_n : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for some $n \geq 1$. We wish to prove that $S_{n+1} : 1 + 2 + \cdots + (n + 1) = \frac{(n+1)(n+2)}{2}$. By the inductive hypothesis, we have that

$$(1+2+\cdots+n)+(n+1) = \frac{n(n+1)}{2}+(n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+2)(n+1)}{2}.$$

Therefore, by the principle of mathematical induction, we have proven the result for all $n \geq 1$.

3. Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all $n \geq 0$.

Solution: First we show the base case. We have that $5^{2\cdot 0+1} + 2^{2\cdot 0+1} = 5 + 2 = 7$, which is divisible by 7. Now assume the inductive hypothesis that $5^{2n+1}+2^{2n+1}$ is true for some $n \geq 0$. Then $5^{2(n+1)+1}+2^{2(n+1)+1} = 5^{2n+1+2}+2^{2n+1+2} = 25\cdot 5^{2n+1}+4\cdot 2^{2n+1} = 21 \cdot 5^{2n+1} + 4(5^{2n+1} + 2^{2n+1})$. The former is divisible by 7 and so is the latter which means the sum is. Thus, by mathematical induction, the result holds for all $n \geq 0$.

1.3 Problems

4. **TRUE** False If we want to prove S_n for all $n \geq 10$, then our base case would be $n = 10$.
5. True **FALSE** When using induction, if we can show that if S_{100} is true, then S_{101} is true, then S_n must be true for all n.

Solution: When doing the inductive step, we must use a general n, not a specific case (although a specific case when help you find a pattern)

6. **TRUE** False Instead of assuming S_n is true and showing that S_{n+1} is true, we can instead assume that S_{n-1} is true and prove that S_n is true.

Solution: This is effectively the same thing in showing that if one day it rains, then it rains the next.

7. Prove that for all $n \geq 1$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Solution: This is true for the base case $n = 1$. Assuming the inductive hypothesis for some $n \geq 1$, we have that

$$\left(\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)}\right) + \frac{1}{(n+1)(n+2)} \overset{IH}{=} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
$$= \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

Therefore, by mathematical induction, the result is true for all $n \geq 1$.

8. Prove that for all $n \geq 1$

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Solution: This is true for the base case and assuming true for some $n \geq 1$, we have that

$$(1 + 4 + \cdots + (3n-2)) + (3(n+1) - 2) \overset{IH}{=} \frac{n(3n-1)}{2} + (3n+1) = \frac{(n+1)(3n+2)}{2}.$$

Therefore, by mathematical induction, we have proven the result for all $n \geq 1$.

9. Prove that

$$1 + 3 + 9 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$$

for all $n \geq 1$.

Solution: The base case is true since $1 = \frac{3-1}{2}$. Then assuming the inductive hypothesis for some $n \geq 1$, we see that

$$1 + 3 + \cdots + 3^n + 3^{n+1} = \frac{3^{n+1} - 1}{2} + 3^{n+1} = \frac{3 \cdot 3^{n+1} - 1}{2} = \frac{3^{n+2} - 1}{2}.$$

Thus, by mathematical induction, the result is shown for all $n \geq 1$.

10. Prove that $6^n - 1$ is divisible by 5 for all $n \geq 1$.

Solution: This is true for the base case $n = 1$. Assuming true for some $n \geq 1$, we have that $6^{n+1} - 1 = 6 \cdot 6^n - 1 = 5 \cdot 6^n + (6^n - 1)$. The former is divisible by 5 and the latter is by the inductive hypothesis, therefore the whole thing is divisible by 5. Thus, by mathematical induction, the result is shown for all $n \geq 1$.

11. Prove that $n^3 + 2n$ is divisible by 3 for all integers $n \geq 0$.

Solution: We show that it is true for the base case $n = 0$. This is true because 0 is divisible by 3. Now we assume the inductive hypothesis that $n^3 + 2n$ is divisible by 3. Now, we have that

$$(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 1 = (n^3 + 2n) + 3(n^2 + n + 1)$$

which is divisible by 3 by the inductive hypothesis. Therefore, we have shown the result by mathematical induction.

12. Let $\{a_n\}_{n\geq 1}$ be a sequence defined as $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$. Prove that $a_n \leq 2$ for all $n \geq 1$.

Solution: Base case: for $n = 1$, $a_1 = 1 \leq 2$.

Inductive step: Suppose that $a_n \leq 2$ for some $n \geq 1$. Then

$$a_{n+1} = \sqrt{a_n + 2} \leq \sqrt{2+2} = 2,$$

so it is also true for $n + 1$. Hence it is true for all n by mathematical induction.

13. Prove that $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \cdots + n! \cdot n = (n + 1)! - 1.$ for all $n \geq 1$.

Solution: The result is true for $n = 1$ and assuming true for general n, we have that

$$1!\cdot 1+\cdots+n!\cdot n+(n+1)!\cdot(n+1) = (n+1)!-1+(n+1)!(n+1) = (n+1)!(n+2)-1 = (n+2)!-1.$$

Thus by mathematical induction, the result is proven.

14. Let $\{a_n\}_{n\geq 1}$ be a sequence defined as $a_1 = 1, a_2 = 5$ and $a_{n+2} = 5a_{n+1} - 6a_n$. Prove that $a_n = 3^n - 2^n$ for all $n \geq 1$.

Solution: Basis case: for $n = 1$, $a_1 = 1 = 3^1 - 2^1$ and for $n = 2$, $a_2 = 5 = 3^2 - 2^2$.

Inductive step: suppose that the statement holds for n and $n+1$. For $n+2$, we have

$$\begin{aligned} a_{n+2} &= 5a_{n+1} - 6a_n = 5(3^{n+1} - 2^{n+1}) - 6(3^n - 2^n) \\ &= 15 \cdot 3^n - 10 \cdot 2^n - 6 \cdot 3^n + 6 \cdot 2^n = 9 \cdot 3^n - 4 \cdot 2^n \\ &= 3^{n+2} - 2^{n+2}, \end{aligned}$$

so it is also true for $n + 2$. Hence it is true for all $n \geq 1$ by mathematical induction.